MATH2050a Mathematical Analysis I

Exercise 4 suggested Solution

12. Show that if $\{x_n\}$is unbounded sequence, then there exists a subsequence such that $\lim 1/x_{nk} = 0$.

Solution:

Since $\{x_n\}$ is unbounded sequence, $\forall k \in N$, there exists $\{x_{n_k}\}$, such that $|x_{n_k}| > k$, we can choose $\{n_k\}$ is a increasing sequence. Hence , we get a subsequence $\{x_{n_k}\}$, satisfies : $1/|x_{n_k}| < 1/k$, for each $k \in N$.

We claim that $\lim 1/x_{nk} = 0$.

for each $\epsilon > 0$, there exists $k_\epsilon \in N$, $\forall k > k_\epsilon$, $|1/k| < \epsilon$. Hence, $\forall k > k_\epsilon$,

$$|1/x_{n_k} - 0| < 1/k < 1/k_\epsilon < \epsilon$$

Therefore, $\lim 1/x_{nk} = 0$.

17. Alternate the terms of the sequences $\{1 + 1/n\}$ and $\{-1/n\}$ to obtain the sequence $\{x_n\}$ given by

(2, -1, 3/2, -1/2, 4/3, -1/3, 5/4, -1/4, ...)

Determine the values of $\limsup x_n$ and $\liminf x_n$. Also, find $\sup x_n$ and $\inf x_n$.

Solution:

Obverse that

$$x_{2k-1} = 1 + 1/k, k \geq 1, \qquad x_{2k} = -1/k, k \geq 1$$

Hence, $\forall i, j \in N$, $x_{2j} < x_{2i-1}$, and $x_{2j} < 0, x_{2i-1} > 1$.

It's easy to check that $\lim x_{2k-1} = 1$, $\lim x_{2k} = 0$. We claim that $\limsup x_n = 1$, $\liminf x_n = 0$. Here we just give the proof of the former one.

For each $\epsilon > 0$, if we can find $k \in N$, $\forall n > k$, we have $x_n < 1 + \epsilon$, on the other hand, since $\lim x_{2k-1} = 1$, then $\limsup x_n = 1$.

Notice that for each $\epsilon > 0$, there exists $k_\epsilon \in N$, $\forall k > k_\epsilon$, $|1/k| < \epsilon$. Hence, $\forall n > 2k_\epsilon + 1$, if n is odd, then there exists m_1, $n = 2m_1 - 1, m_1 > k_\epsilon$, $x_n = 1 + \frac{1}{m_1} < 1 + \frac{1}{k_\epsilon} < 1 + \epsilon$. if n is even, then there exists m_2, $n = 2m_2, m_2 \geq k_\epsilon$, $x_n = -\frac{1}{m_2} < 1 < 1 + \epsilon$. To conclude, $\forall n > 2k_\epsilon + 1$, $x_n < 1 + \epsilon$, completing the proof.

3. Show directly from the definition that the following are not Cauchy sequence.

(a) $\{(-1)^n\}$ (b) $\{n + \frac{(-1)^n}{n}\}$ (c) $\{lnn\}$

Solution:

(a) Fix $\epsilon_0 = \frac{1}{2}$, $\forall k \in N$, we can choose $n = 2k$, $m = 2k+1$, $|(-1)^n - (-1)^m| = 2 > \frac{1}{2}$. Hence, $\{(-1)^n\}$ is not a Cauchy sequence.

(b) Fix $\epsilon_0 = 1$, similarly, $\forall k \in N$, we can choose $n = 4k$, $m = 2k$, $|x_n - x_m| = |4k + \frac{1}{4k} - 2k - \frac{1}{2k}| = |2k - \frac{1}{4k}| > 2k - 1 \geq 1$.

(c) Fix $\epsilon_0 = \frac{1}{2}ln2$, $\forall k \in N$, choose $n = 4k$, $m = 2k$, $|x_n - x_m| = |ln4k - ln2k| = ln2 > \epsilon_0$.

5. If $x_n = \sqrt{n}$, show that $\{x_n\}$ satisfies $\lim|x_{n+1} - x_n| = 0$, but that it is not a Cauchy sequence.

Solution:

Since $(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n}) = 1$, we have

$$|x_{n+1} - x_n| = |\sqrt{n+1} - \sqrt{n}| = |\frac{1}{\sqrt{n+1} + \sqrt{n}}|$$

Hence, $0 \leq |x_{n+1} - x_n| \leq \frac{1}{\sqrt{n}}$. Since $\lim \frac{1}{\sqrt{n}} = 0$, we have $\lim|x_{n+1} - x_n| = 0$. Next we prove $\{x_n\}$ is not a Cauchy sequence. Fix $\epsilon_0 = \frac{1}{2}$, $\forall k \in N$, choose $n = (k+1)^2$, $m = (k)^2$, $|x_n - x_m| = 1 > \epsilon_0$. Therefore, it is not a Cauchy sequence.